Mead

COMPOSITION

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LOGBOOK #63

27 APR 2000 → 06 JUNE 2000

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EMBRACING COMPLEXITY

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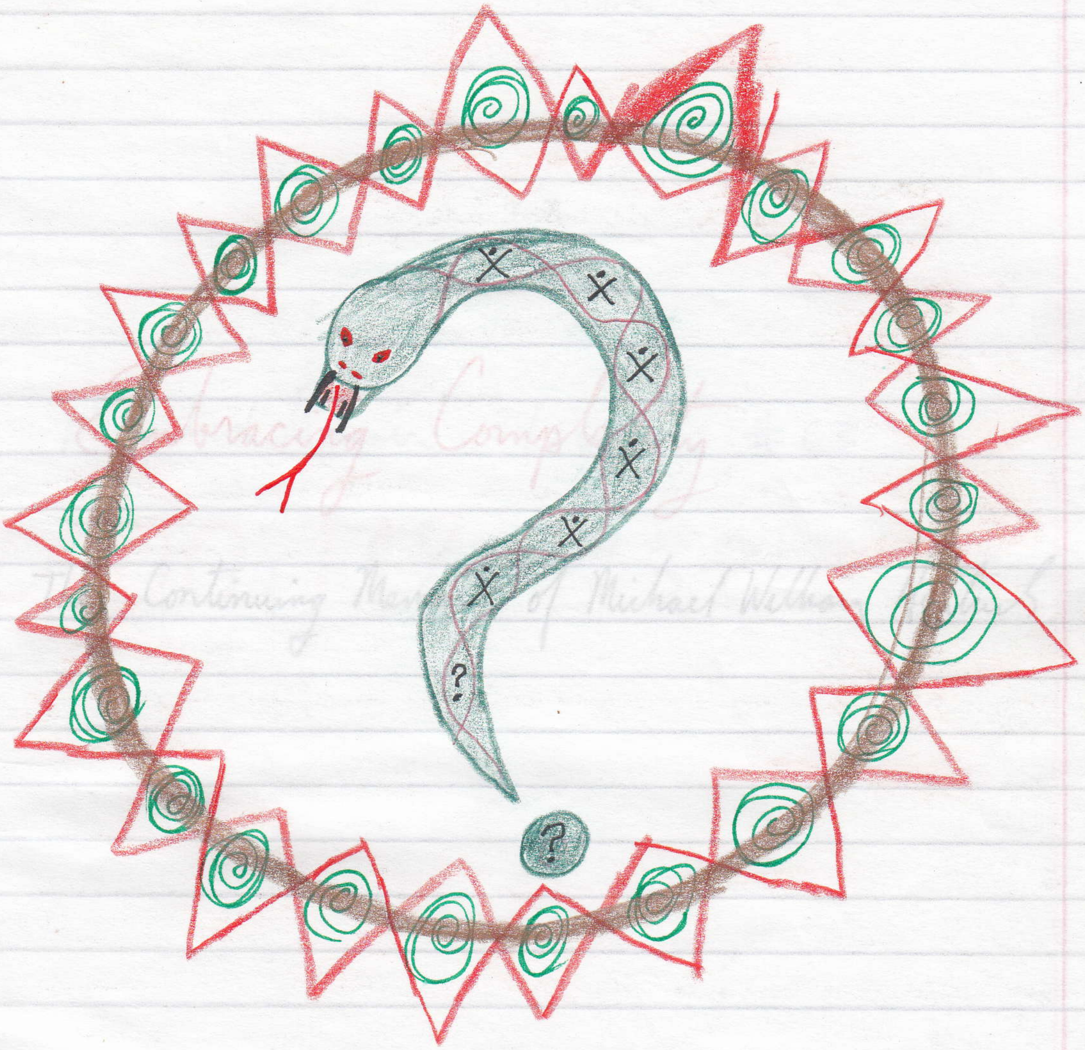

Embracing Complexity

The Continuing Memoirs of Michael William Hentrich

Zone One

Cosmogonic Discomfort

"All is suffering" — modernized, the Buddhist expression runs: "All is nightmare." Every individual discomfort leads back, ultimately, to a cosmogonic discomfort, each of our sensations expiating that crime of the primordial sensation, by which Being crept out of somewhere

— E. M. Cioran
"The Trouble With Being Born"

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All is nightmare, but there are varying degrees of nightmare. There are the nightmarish pressures of final exams at a high caliber University, and there are the nightmarish immediate demands of relentless chemical dependence. There are degrees of nightmare; physical addiction!

Still, every single individual discomfort does indeed lead back, ultimately, to a cosmogonic discomfort. Even the stress from the pressures of final exams expiate that crime of the primordial sensation, by which being crept out of somewhere.

As it is after midnight, I am going to give myself some SLACK. All I will force myself to do is reread section 7.1 of the Linear Algebra text book: Linear Transformations. That is all. Oh, yes — and, also, on page 2 herein I want to list the Master Plan with a Part Time Slacker Alternative Plan as well.

119. 1500 I have to be satisfied with the work I did (M250). I was up into the night, and, even though I rose late this afternoon, I still went ahead and barrelled through 7.2 problems. I will skip 7.3 problems (not essential).

So, I will be devoting myself to M251 & M300 in tΣ for the next two days. I will have another M250 session on Monday after the review.

I will be leaving for Freehold shortly, but first I will take some notes from Internet on Green's Theorem into tΣ.

2130 Neither Levitt's notes nor text book is any help with verifying Green's Theorem. I hope Chris Long can help us next weekend. I have put too much time into this Calculus 3 class. I will look for info at Barnes and Noble, but then I have to start pecking away at M300 tΣ meditations.

2200 I will try to translate what I read into Levitts notation,

where $d\vec{r} = \langle x'(t), y'(t), z'(t) \rangle = \vec{T}\,|ds|$

because $\vec{T}(t) = \dfrac{\langle x'(t), y'(t), z'(t) \rangle}{\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}}$

and $|ds| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$ so $|ds|$ cancels out the denominator of $\vec{\tau}(t)$

But, instead of using the conventional $d\vec{r}$, I will use Norman Levitt's notation, $\vec{T}\,|ds|$ realizing all the while this is simply $\left\langle \frac{dx}{dt}, \frac{dx}{dt}, \frac{dz}{dt} \right\rangle$, which will be

dotted with $\vec{F} = P(x(t), y(t), z(t))\,dx + Q(x(t), y(t), z(t))$ etc.

Green's Theorem

$$\oint_C [P\,dx + Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

To be independent of path,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

I know in my gut that this is the exact same thing as $f_{xy} = f_{yx}$, $f_{zx} = f_{xz}$, $f_{yz} = f_{zy}$

Green's Theorem in Space (Divergence Theorem)

$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,dS$$

can also be written as

$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV = \iint_S [P\,dy\,dz + Q\,dz\,dz + R\,dx\,dy]$$

In vector form, with $\vec{F} = \langle P, Q, R \rangle$

and $n = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$ these can be simply written as

$$\iiint \vec{\nabla} \cdot \vec{F}\,dV = \iint_S \vec{F} \cdot n\,dS$$

13

121. 1730. I am presently at Border's Bookstore in New Brunswick. I found a few books on Advanced Calculus, namely Green's Theorem. Rather than fill the last remaining pages of Techgnostic Scribblings, volume one, with notes on theory (I want to save those last pages for worked out sample problems), I will take some notes herein. I may use Philo so as to keep it neat.

Note page 9 about $\vec{T}(t)$

$\vec{T}(t)$ is the normal tangent vector defined as defined on page 9. We normalize $\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$

Note that when we see dx, dy, dz, we read $x'(t)\,dt$, $y'(t)\,dt$, $z'(t)\,dt$

so that $\int \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot r'(t)\,dt$

$$\int \vec{F} \cdot \vec{T}\ |ds| = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{\|\vec{r}(t)\|}\ \|\vec{r}(t)\|\ dt$$

$$= \int_a^b \left(\vec{F}(\vec{r}(t)) \cdot \vec{T}(t) \right) \|r'(t)\|\ dt$$

$$= \int_a^b \left(\vec{F} \cdot \vec{T} \right) |ds|$$

Is this also the line integral?

$|ds|$ will always mean $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T}\, |ds|$$

$$= \int_a^b \left(P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k} \right) \cdot \left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \right) dt$$

$$= \int_a^b \left[P(x,y,z)\frac{dx}{dt} + Q(x,y,z)\frac{dy}{dt} + R(x,y,z)\frac{dz}{dt} \right] dt$$

This is $\int_C P\,dx + Q\,dy + R\,dz$

Where $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ for $0 \le t \le 1$

$x = t, \quad y = t^2, \quad z = t^3$

$dx = 1\,dt \quad dy = 2t\,dt, \quad dz = 3t^2\,dt$

so $\int_C (y+z)\,dx + (x+z)\,dy + (x+y)\,dz$

$$= \int_0^1 (t^2 + t^3)\,dt + (t + t^3)\,2t\,dt + (t + t^2)\,3t^2\,dt$$

$$= \int_0^1 t^2 + t^3 + 2t^2 + 2t^4 + 3t^3 + 3t^4 \; dt$$

$$= \int_0^1 3t^2 + 4t^3 + 5t^4 \; dt$$

$$= t^3 + t^4 + t^5 = 3$$

Where does $\int \vec{F} \cdot |ds|$ come in?

What Green's Theorem does is relate the VECTOR LINE INTEGRAL around a closed curve in R^2 ~~with~~ to an appropriate ~~a~~ double integral over the plane region R ~~one~~ bounded by the closed curve C.

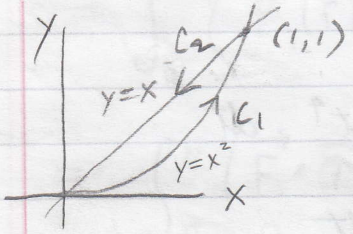
The fact that there is such an elegant connection between one and two dimensional integrals is at once surprising, satisfying, and powerful. Green's Theorem is, as stated before:

Let $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ be a vector field, then

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{dx} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Example:

Let $\vec{F} = xy\,\vec{i} + y^2\,\vec{j}$ be the first quadrant region bounded by the line $x=y$ and the parabola $y=x^2$. We verify Green's Theorem in this case.



$$\oint_C \vec{F}\cdot\vec{T}\,|ds| = \oint_C xy\,dx + y^2\,dy$$

$$C_1 : \begin{cases} x = t \\ y = t^2 \end{cases} \quad 0 \le t \le 1 \qquad \begin{aligned} dx &= dt \\ dy &= 2t\,dt \end{aligned}$$

(1,1) to (0,0)

so $x(t) = 1 + (0-1)t$

$y(t) = 1 + (0-1)t$

$$C_2 : \begin{cases} x = 1-t \\ y = 1-t \end{cases} \quad 0 \le t \le 1 \qquad \begin{aligned} dx &= -dt \\ dy &= -dt \end{aligned}$$

$$\oint_C xy\,dx + y^2\,dy = \int_{C_1} xy\,dx + y^2 dy$$

$$+ \int_{C_2} xy\,dx + y^2\,dy$$

$$= \int_0^1 (t^3\,dt) + (t^4)(2t\,dt) + \int_0^1 (1-t)^2(-dt) + (1-t)^2(-dt)$$

$$= \int_0^1 t^3 + 2t^5\,dt + \int_0^1 2(1-t)^2(-dt)$$

$$= \left[\frac{t^4}{4} + \frac{t^6}{3}\right]_0^1 + \left[\frac{2(1-t)^3}{3}\right]_0^1$$

$$= \left[\left(\frac{1}{4}+\frac{1}{3}\right) - 0\right] + \left[(0) - \left(+\frac{2}{3}\right)\right] = \frac{7}{12} - \frac{8}{12} = -\frac{1}{12}$$

That was DIRECTLY, now, by $\iint_R \left[\underset{\downarrow\ \emptyset}{\frac{\partial}{\partial x}(y^2)} - \underset{\downarrow\ -x}{\frac{\partial(xy)}{\partial y}}\right] dx\,dy$

$$= \int_0^1 \int_{x^2}^{x} -x\,dy\,dx$$

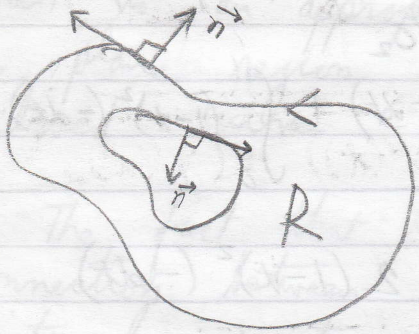
$$= \int_0^1 -x(x-x^2)\,dx = \int_0^1 (x^3 - x^2)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1 = \frac{1}{4} - \frac{1}{3} = \frac{3}{12} - \frac{4}{12} = \frac{-1}{12}$$

The line integral and the double integral agree, just as Green's Theorem says they must.

Quick note about DIVERGENCE THEOREM:

$\vec{n}$ is the outward normal vector to R



and $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$

then $\oint_C \vec{F} \cdot \vec{n}\, ds$

$$= \iint_R \vec{\nabla} \cdot \vec{F}\, dA$$

ALTERNATIVE FORMULATIONS

We can rewrite the line integral / double integral formula in two ways.

These reformulations generalize to higher dimensions and provide some additional insight in interpreting the geometric significance of Green's Theorem.

Consider vector field $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$

The "curl" of $\vec{F}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix}$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$? \quad \vec{i}\left(0 - \frac{\partial Q}{\partial z}\right) - \vec{j}\left(0 - \frac{\partial P}{\partial z}\right) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

so $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R (\vec{\nabla} \times \vec{F}) \cdot \vec{k} \, dA$

since $\oint_C \vec{F} \cdot d\vec{r} = \oint_C P\,dx + Q\,dy$

Note that $\vec{n} = \dfrac{y'(t)\,\vec{i} - x'(t)\,\vec{j}}{\sqrt{(x'(t))^2 + (y'(t))^2}}$

$$= \frac{y'(t)\,\vec{i} - x'(t)\,\vec{j}}{\| \vec{r}'(t) \|} \longleftarrow |ds|$$

1930. OK, that's enough.
I will return to my cave in Highland Park, drink coffee, and go over Mathematical Reasoning before any more Linear Algebra or Calculus; advanced calculus that is . . .

121. 2030 It looks as though $\text{div}\, \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

and $\text{curl}\, \vec{F} = \vec{\nabla} \times \vec{F}$

Recall that the terribly mysterious $|ds|$ is simply

$$|r'(t)\,dt| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

then distance travelled is $\int_a^b |r'(t)\,dt|$

and the arc length is $s(t) = \int |r'(t)|\,dt$

$$\frac{ds}{dt} = |r'(t)|$$

The arc length of the curve is $|ds| = |r'(t)\,dt|$

The Fundamental Theorem for Line Integrals
(Analogous to the Fundamental Theorem of Calculus)

First, let us state the fundamental theorem of calculus:

Defn: $F(t) = \int_a^t f(x)\,dx$

$F(t)$ is simply the area under the $f(x)$ curve from a to t.

$$F'(t) = f(t)$$

Let $F(t) = \int f(t)\,dt$

then $F'(t) = f(t)$

Version 2: $F(t) = F(a) + \int_a^t F'(x)\,dx$

$$F(t) = \int F'(t)\,dt + C$$

So, for Line Integrals:

$$\int \vec{\nabla} f \cdot d\vec{r} = f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))$$

$$\vec{\nabla} f = \vec{F} = \langle P, Q, R \rangle$$

$$f_x = P(x, y, z)$$
$$f_y = Q(x, y, z)$$
$$f_z = R(x, y, z)$$

Determine whether or not $F(x,y) = \langle 1+4x^3y^3, 3x^4y^2 \rangle$ is a conservative vector field. If it is, find a potential.

$$\vec{F} = \vec{\nabla} f = \langle f_x, f_y \rangle$$

$$f_x = 1 + 4x^3y^3 \qquad f_y = 3x^4y^2$$

We integrate to determine f.

$$f(x,y) = \left[\int f_x(x,y)\,dx\right] + \gamma(y)$$

$$= \left[\int 1 + 4x^3y^3\,dx\right] + \gamma(y)$$

$$= x + x^4y^3 + \gamma(y)$$

We then differentiate this f w.r.t y

$$f_y(x,y) = 0 + 3x^4y^2 + \gamma'(y)$$

We substitute the original f_y to find

$$3x^4y^2 = 3x^4y^2 + \gamma'(y)$$

so $\gamma'(y) = 0$, so $\gamma(y) = k$

and $f(x,y) = x^4y^3 + x + k$

note that $f_{xy} = 12x^3y^2 = f_{yx} = 12x^3y^2$

so we knew from the start that $\vec{F} = \vec{\nabla} f$ was a vector field.

121.2130 I promised I would not get too caught up with advanced calculus work, and yet I have. For the rest of the night – just M300 stuff. One final note about line integrals – Sometimes it is w.r.t x and y or just s.

First step is to split into line segments, parameterize the variables and write in terms of dt.

Example: if $\int_C xy\,dx + (x-y)\,dy$, and segment is from $(2,0)$ to $(3,2)$

$$x(t) = 2 + (3-2)t = 2+t$$
$$y(t) = 0 + (2-0)t = 2t$$
$$0 \le t \le 1$$

so $dx = 1\,dt$ and $dy = 2\,dt$

so $$\int_C (2+t)(2t)\,dt + \left[(2+t)-2t\right]2\,dt$$

$$= \int_C \left(4t + 2t^2 + (4-2t)\right) dt$$

$$= \int_0^1 2t^2 + 2t + 4\,dt = \frac{2t^3}{3} + \frac{2t^2}{2} + 4t \Big]_0^1$$

$$= \frac{2}{3} + 1 + 4 = \frac{5}{3} + \frac{12}{3} = \frac{17}{3}$$

But if given $\int_C xyz\,ds$, after parameterizing

$r(t) = \langle x(t), y(t), z(t) \rangle$, $|ds| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t)}$

We can also evaluate by $\int_C \vec{F} \cdot d\vec{r}$ and $\int_C \vec{F} \cdot \vec{T}\,|ds|$

I have reviewed this enough. I am prepared for the recitation with TA. NOW: MEMORIZE M300 t S session!

$\boxed{\begin{matrix}2000.122.1\\05.01.0030\end{matrix}}$ I feel prepared to take the M300 final exam, and it is not until Thursday 5/4 at 8PM. So I feel at ease with my "grid". I will study Linear Algebra after the 250 review session tomorrow from 2PM to 7PM; then, I will eat chow before M251 recitation. From 9PM until after midnight I will continue to study advanced calculus.

On Tuesday I will bring ±$ and Calculus Sketch diary to work, with mechanical pencil... When I return I will quickly go over M300 ±$ (and Set Theory); but, if I can, I also want to peck away a little at M250 and M251 as I do feel quite prepared for M300. I will be going with the flow.

Let's look at $n^3 = 6\binom{n}{3} + 6\binom{n}{2} + n$

Is not "$6\binom{n}{3}$" a permutation?

Permutations take into account ORDER. Ordering the arrangement makes a huge difference.

$$P(n,k) = \frac{n!}{(n-k)!} \quad \text{so} \quad P(n,\cancel{k}3) = \frac{n!}{(n-3)!}$$

example $P(10,4) = \frac{10!}{6!} = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$

example $P(4,3) = \frac{4!}{\cancel{1}!} = 4 \cdot 3 \cdot 2 = 24$

Same as $k!\binom{n}{k} = k! \dfrac{n!}{k!(n-k)!}$

$3!\binom{4}{3} = 24$

Combinations don't account for orderings.

$$C(n,k) = \frac{P(n,k)}{P(k,k)}$$

As for studying for exams, the heat is getting to me. I may go through the concepts to be covered right here in L63 in pencil. This will keep me focused. Even if I can endure the heat up in this attic room over the summer, I am not sure if my computer will be able to handle it.

After the cold shower, my body temperature went down a little. It is difficult for me to believe that one day I will be earning enough money to be able to afford to dwell in a climate controlled environment. These days will have made me stronger.

Even if my landlady were to allow me to put a window unit (AC) in this oven room, I would not want to take my AC unit back from my sister as they are using it in their daughter's room. I would not want to be "the one who took away "their air conditioner"".

So... while I work on my machine over the next week, I will also be going over math problems, developing my QUANTITATIVE SKILLS.

My practical computer-user oriented skills are maturing as I gain experience in losing files. Important data must be stored on ~~secondary~~ media other than secondary storage if it is to be preserved. Nothing beats a hard copy in a LOGBOOK!

127. 2245 Right Here, Right Now SESSION... As tZONE" is full, I will go over some crucial concepts for both Multivariable Calculus and Linear Algebra right here in L63 over the next week. I will not begin a new theme until after the last final exam (afternoon 5/11). Then I will begin a new theme because I will be in a new psychological, electro-chemical ZONE. Now... math.

Final Concept for Multivariable Calculus, 640:251:
The Divergence Theorem — An example.

$\vec{F} = xy^2\,\vec{i} + x^2y\,\vec{j}$ $\qquad R = \{x^2+y^2 \leq 1\}$

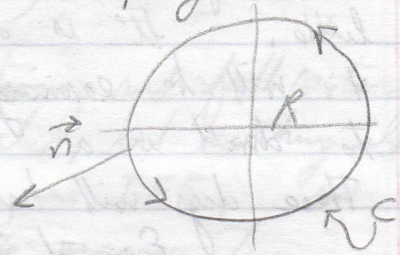
$\downarrow$

$P(x,y) = xy^2$

$Q(x,y) = x^2y$

does not really come into play.

Parameterization:



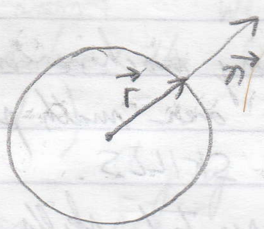
$x = r\cos\theta = \cos\theta$

$y = r\sin\theta = \sin\theta$

$0 \leq \theta \leq 2\pi$

Think: What is $\vec{n}$ at the point (x,y) on the curve? $\vec{n}(x,y) = x\,\vec{i} + y\,\vec{i} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$

Why?



because $\left(\frac{dy}{dt} + \frac{-dx}{dt}\right)$

so $\vec{F} = \cos\theta\sin^2\theta\,\vec{i} + \cos^2\theta\sin\theta\,\vec{j}$

$\vec{F}\cdot\vec{n} = \langle xy^2, x^2y\rangle \cdot \langle x, y\rangle = x^2y^2 + x^2y^2$
$= 2x^2y^2$

$= 2\cos^2\theta\sin^2\theta$

So we compute $\oint_C \vec{F}\cdot\vec{n}\,|ds|$

first $\frac{dx}{d\theta} = -\sin\theta$ and $\frac{dy}{d\theta} = \cos\theta$

so $ds = \sqrt{\sin^2\theta + \cos^2\theta} = \sqrt{1} = 1$

$\oint_C \vec{F}\cdot\vec{n}\,|ds|$ awaits, but let's not be too hasty. Simplify $\vec{F}\cdot\vec{n}$ first:

$2\cos^2\theta \sin^2\theta$

Think: $2\sin\theta\cos\theta = \sin(2\theta)$

How so? I remember the formula as follows:

$$\cos^2\theta + \sin^2\theta = 1$$
$$\cos^2\theta - \sin^2\theta = \cos(2\theta)$$

adding $\longrightarrow 2\cos^2\theta = 1 + \cos(2\theta) \rightarrow \cos^2\theta = \frac{1}{2} + \frac{\cos(2\theta)}{2}$

subtracting $\longrightarrow 2\sin^2\theta = 1 - \cos(2\theta) \rightarrow \sin^2\theta = \frac{1}{2} - \frac{\cos(2\theta)}{2}$

but the formula we need here is before all that, having to do with a double angle formula, so ignore the other one. It is still too useful to be erased.

So, square $\sin(2\theta) = 2\sin\theta\cos\theta$

$\rightarrow \sin^2(2\theta) = 4\sin^2\theta\cos^2\theta$

This is twice what we need: $2\cos^2\theta\sin^2\theta$ so we just use $\frac{\sin^2(2\theta)}{2}$.

Now, $\oint_C \vec{F}\cdot\vec{n}\,|ds| = \frac{1}{2}\oint_C \sin^2(2\theta)\,d\theta$

(and $|ds| = 1$) But we need the other formulas as well $\sin^2(2\theta) = \frac{1}{2} - \frac{\cos(4\theta)}{2}$

$$\oint_C \vec{F}\cdot\vec{n}\,|ds| = \frac{1}{2}\int_0^{2\pi} \frac{1}{2} - \frac{\cos(4\theta)}{2}\,d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} 1 - \cos(4\theta)\,d\theta = \frac{1}{4}\left(\theta - \frac{\sin(4\theta)}{4}\right)\Bigg]_{\theta=0}^{\theta=2\pi}$$

$$= \frac{1}{4}\Big[(2\pi - 0) - (0 - 0)\Big] = \frac{2\pi}{4} = \frac{\pi}{2}$$

THAT WAS with $\oint \vec{F}\cdot\vec{n}\,|ds|$
NOW we will use the Divergence Theorem

Using the divergence theorem simplifies this problem tremendously. It is very powerful.

$$\oint \vec{F} \cdot \vec{n} \, |ds| = \iint_R y^2 + x^2 \, dA$$

Why? Because the divergence theorem says that $\oint_C \vec{F} \cdot \vec{n} \, |ds| = \iint_R P_x + Q_y \, dA$

remember $P(x,y) = xy^2 \longrightarrow P_x = y^2$
$Q(x,y) = yx^2 \longrightarrow Q_y = x^2$

so we compute with $0 \leq \theta \leq 2\pi$
$0 \leq r \leq 1$

but, life gets even simpler!

$x^2 + y^2 = r^2$ and our area integral becomes $\int_0^{2\pi} \int_0^1 r^2 \, r \, dr \, d\theta$

remember, when translating to polar coordinates, $dA \mapsto r \, dr \, d\theta$

$$\int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta = \int_0^{2\pi} \frac{r^4}{4} \Big[_{r=0}^{r=1} \, d\theta$$

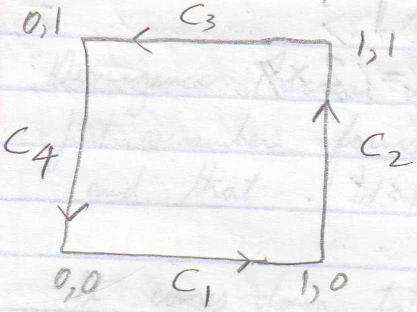
$$\int_0^{2\pi} \frac{1}{4} \, d\theta = \frac{\theta}{4} \Big[_{\theta=0}^{\theta=2\pi} = \frac{2\pi}{4} = \frac{\pi}{2}$$

and the divergence theorem is verified.

Now, an example of Green's Theorem:

$$\oint_C \vec{F} \cdot \vec{\tau}\, |ds| = \iint_R Q_x - P_y \, dA$$

let $\vec{F} = y\,\vec{i} + x^2 y\,\vec{j}$



$\oint_C \vec{F} \cdot \vec{\tau}\, |ds| = \int P\,dx + Q\,dy$

C_1: $(0,0)$ to $(1,0)$ $\quad 0 \le t \le 1$

$\vec{\tau} = +\vec{i}$

$x = t \quad \frac{dx}{dt} = 1\,dt$

$y = 0 \quad \frac{dx}{dt} = 0$

$\int_0^1 0 + t^2(0)(1)\,dt$

$= \int_0^1 0\,dt = 0$

C_2: $(1,0)$ to $(1,1)$

$\vec{\tau} = +\vec{j}$

$x = 1 + (1-1)t = 1 \quad \frac{dx}{dt} = 0$

$y = t \quad \frac{dy}{dt} = 1\,dt$

$\int_0^1 t(0) + 1(t)(1)\,dt = \int_0^1 t\,dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$

C_3: $(1,1)$ to $(0,1)$

$\vec{\tau} = -\vec{i}$

$x = 1 + (0-1)t = 1 - t \quad \frac{dx}{dt} = -1\,dt$

$y = 1 \quad \frac{dy}{dt} = 0$

$\int_0^1 1(-1)\,dt + (1-t)^2(1)(0)\,dt = \int_0^1 -dt = -t\Big|_0^1 = -1$

C_4: $(0,1)$ to $(0,0)$ $\quad x = 0 \quad \frac{dx}{dt} = 0$

$y = 1 - t \quad \frac{dy}{dt} = -1\,dt$

$\int_0^1 (1-t)(0)\,dt + (0)^2(1-t)\,dt$

$= 0$

so $\int P\,dx + Q\,dy = 0 + \frac{1}{2} - 1 + 0 = -\frac{1}{2}$

NEXT: Verify using Green's Theorem.

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Green's Theorem says that $\oint_C \vec{F} \cdot \vec{T} \, |ds|$

$$= \iint_R Q_x - P_y \, dA$$

Here, $P(x,y) = y \longrightarrow P_y = 1$

$Q(x,y) = x^2 y \longrightarrow Q_x = 2xy$

so $\int_0^1 \int_0^1 2xy - 1 \, dy \, dx$

$$= \int_0^1 \frac{2xy^2}{2} - y \Big[_{0}^{y=1} dx$$

$$= \int_0^1 (x-1) \, dx = \frac{x^2}{2} - x \Big[_{x=0}^{x=1}$$

$$= \left(\frac{1}{2} - 1\right) - (0) = -\frac{1}{2}$$

Verified, and much easier to compute.

2000.128.0
05.07.0030

After Netscape 6 PR1 is installed (I am saving the downloaded files onto LS-120 Holy Files 2 as well as H:LAB on second disk) I will do a system-save before installing TweakUI, Plus!, MatLab, and the compilers.

I really do not want to put in IE5. IE4 suits me fine. I dont want any conflicts. If anything, I will also set up Netscape Communicator 4.7. I will be awake most of the night, as I am in a good frame of mind for math. I am working it out while supervising ~~the~~ myself rebuilding my system. I lost all my links for the internet. I will have to find out how to prevent this. (SAVE USER. PROFILE).

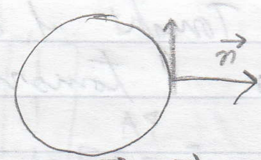
Rather than to keep writing, I will try to go through tS sessions mentally, starting from the most recent stuff, working backwards.

Checklist: MC → Multivariable Calculus
LA → Linear Algebra

MC Divergence Theorem ✓ Green's Theorem ✓ Line Integrals: just remember forms $\int \vec{F}\, ds$, $\int \vec{F} \cdot \vec{T}\, |ds|$, $\int \vec{F} \cdot d\vec{r}$ and that $|ds| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

It comes down to $\int P\,dx + Q\,dy + R\,dz$.
The only thing is that I am still uncertain exactly how $\vec{n}$ is derived for divergence... I know it is the normal to the curve.

area → $x^2 + y^2 = 1$

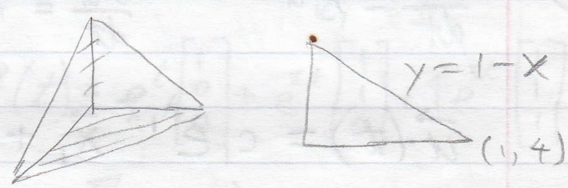


I don't know where $\vec{n} = x\vec{i} + y\vec{j}$ comes from, but I am sure it is from geometry.

In 3-space, where $\vec{F} = x^2\vec{i} - 2xy\vec{j} + 10z\vec{k}$ for the region bounded by $2x + 2y + z = 2$ and xy, xz, yz planes: $P = x^2$, $Q = -2xy$, $R = 10z$

$\nabla \vec{F} = P_x + Q_y + R_z = 2x - 2x + 10 = 10$

$\iiint_V 10\, dV$



for $\iint \vec{F} \cdot \vec{n}\, |ds|$ of other problem, why is $\vec{n} = x\vec{i} + y\vec{j}$
Well, $\vec{n} \perp \vec{T}$, and $r'(t) = $???

I will focus on it tomorrow. Now, at 0300, I am sleepy. I will set up Treak, then do a SYSTEM SAVE.

128.1220 Setting up MATLAB this morning, I realize why I could not go back into diary file after closing it. I had never set up notebook!
How? At MATLAB EDU>> prompt, type notebook -setup and follow directions.
Also, to set up a compiler to use with MATLAB, mex -setup, choose Borland C.

Although I will go through some concepts for Calc3 exam later tonight, this afternoon, I want to tighten up on Linear Algebra.

128.1400 I know it is best to go over M250 this afternoon. Tonight I can tighten up on M251, as well as tomorrow before the exam.

① $PA = LU$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix} \longrightarrow \overset{PA}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{bmatrix}} \longrightarrow l_{31} = 2 \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 7 \end{bmatrix} \longrightarrow l_{32} = 3 \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

So $PA = LU = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}$

② $\frac{d\vec{u}}{dt} = A\vec{u}$ $\qquad \frac{d\vec{u}}{dt} = \lambda \vec{u}$ has solutions $\vec{u}(t) = Ce^{\lambda t}$

$$\vec{u}(t) = c_1 e^{\lambda_1 t}\vec{x}_1 + \dots + c_n e^{\lambda_i t}\vec{x}_n$$

So, if I have found eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

Solve $\frac{d\vec{u}}{dt} = A\vec{u}$ starting from $\vec{u}(0) = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$

just 3 steps, my friend.

Overview: STEP 1: find eigenvectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$
STEP 2: solve for $\vec{c}$: $\vec{x}\vec{c} = \vec{u}(0)$
STEP 3: $\vec{u}(t) = c_1 e^{\lambda_1 t}\vec{x}_1 + \dots + c_3 e^{\lambda_3 t}\vec{x}_3$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

eigenvectors for $\lambda_1 = 1$
$(A - \lambda_1 I)\vec{x} = \vec{0}$ etc

$$\lambda_1: \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \vec{x}_1 \end{bmatrix} = \vec{0} \qquad \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2: \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \vec{x}_2 \end{bmatrix} = \vec{0} \qquad \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3: \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \vec{x}_3 \end{bmatrix} = \vec{0} \qquad \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Now, once we have the eigenvectors, we can solve for $\vec{c}$ as follows: $\vec{x}\vec{c} = \vec{u}(0)$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$$

use back substitution: $c_3 = 4$, $c_2 = 1$, $c_1 = 1$

We write, quite simply, $\vec{u}(t) = e^{t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e^{2t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 4e^{3t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

That's all there is to it. There is no more.

Now, just in case we have to deal with second order equations of the form $y'' + by' + ky = 0$ we let $y = e^{\lambda t}$ and solve $(\lambda^2 + b\lambda + k)e^{\lambda t}$
Now it all depends on $\lambda^2 + b\lambda + k = 0$

45

$\frac{dy}{dt} = y' \quad \frac{dy'}{dt} = -ky - by'$

converts to $\frac{d}{dt}\begin{bmatrix} y \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -k & -b-\lambda \end{vmatrix} = \lambda^2 + b\lambda + k = 0$$

The eigenvectors and the complete solution:

$\vec{x}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \quad \vec{x}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$

$$\vec{u}(t) = c_1 e^{\lambda_1 t}\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t}\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

example: from $y'' + 4y' + 3y = 0$

$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \quad \frac{d\vec{u}}{dt} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\vec{u} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} y \\ y'' \end{bmatrix}$

y'
$-3y - 4y'$

Don't go through the entire $\det(A - \lambda I)$ process. Just use $\lambda^2 + 4\lambda + 3 = 0$ $\quad \lambda_1 = -1$
$(\lambda+1)(\lambda+3) = 0 \quad \lambda_2 = -3$

trace: $0 - 4 = -4 = \lambda_1 + \lambda_2 = -4$ ✓
det: $0 - -3 = 3 = \lambda_1 \lambda_2 = 3$

eigenvectors $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

no problem,

$$\vec{u}(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

(3) Now, how to go about not confusing $y'' + by' + ky = 0$ with $e^{At} = Se^{\Lambda t}S^{-1}$?
The solution looks the same, but

$$e^{At}\vec{u}(0) = Se^{\Lambda t}S^{-1}\vec{u}(0) = \begin{bmatrix} x_1 \cdots x_n \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

1. find λ's and $\vec{x}$'s
2. Write $\vec{u}(0) = \vec{c}_1\vec{x}_1 + \ldots + \vec{c}_n\vec{x}_n$
3. Multiply each $\vec{x}_i$ by $e^{\lambda_i t}$

(4) $AS = S\Lambda \longrightarrow S^{-1}AS = \Lambda \longrightarrow A = S\Lambda S^{-1}$
$A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}$

Matrix Powers
1. Find λ's and $\vec{x}$'s
2. Write u_0 as above
3. Multiply each $\vec{x}_i$ by $(\lambda_i)^k$

then $\vec{u}_k = A^k u_0 = c_1(\lambda_1)^k\vec{x}_1 + \ldots + c_n(\lambda_n)^k\vec{x}_n$

A^k is $(S\Lambda S^{-1})^k = S\Lambda^k S^{-1}$

$u_0 = S\vec{c}$

An example: $\frac{d\vec{u}}{dt} = A\vec{u}$ with $A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$

and $\vec{u}(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ trace $\rightarrow -5$ $\lambda_1 = 0$
det $\rightarrow 0$ $\lambda_2 = -5$

$\lambda_1 = 0$: $x_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\lambda_2 = -5$: $\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$S = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$ $\det(S) = -5$ $C_S = \begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}$; $C_S^T = \begin{bmatrix} -1 & -1 \\ -2 & 3 \end{bmatrix}$

$S^{-1} = \frac{1}{-5}\begin{bmatrix} -1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{bmatrix}$; $A = \underbrace{\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}}_{S}\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}}_{\Lambda}\underbrace{\begin{bmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{bmatrix}}_{S^{-1}}$

Once we find $A = S\Lambda S^{-1}$, we can solve for $\vec{u}(t)$ from given $\vec{u}(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

Solve $S\vec{c} = \vec{u}(0)$

$$\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \qquad \vec{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\vec{u}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 2e^{-5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

notice that $c_1 = 1$ and $\lambda_1 = 0$ so $e^{0t} = 1$

Now we can compute the matrix e^{At} using A and Λ.

$$A = S\Lambda S^{-1} \longrightarrow e^{At} = S e^{\Lambda t} S^{-1}$$

so $e^{\Lambda t} = \begin{bmatrix} e^0 & 0 \\ 0 & e^{-5} \end{bmatrix}$ no problem

and $e^{At} = \underbrace{\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & e^{-5t} \end{bmatrix}}_{e^{\Lambda t}} \begin{bmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{bmatrix}$

When $\lambda < 0$, $e^{\lambda t} \to 0$ $\qquad e^{-5t} \to 0$
When $\lambda > 0$, $e^{\lambda t} \to \infty$

so, as $t \to \infty$ $\vec{u}(t) \to \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

and $e^{At} \to \frac{1}{5} \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$

(5) I have to go over Similar Matrices:

$\det(M) \neq 0 \rightarrow A$ and $M^{-1}AM$ are similar

if $B = M^{-1}AM$, then $A = MBM^{-1}$

A diagonal matrix is similar to Λ.
In that special case M is S.

Just remember connections between A and $B = M^{-1}AM$

Some things change	Some do not change
eigenvectors $\vec{x} \rightarrow M^{-1}\vec{x}$	eigenvalues
nullspace	trace
columnspace	determinant
row space	rank
left nullspace	number of independent eigenvectors
singular values σ	Jordan Form

(6) $T(c\vec{v} + d\vec{w}) = cT(\vec{v}) + dT(\vec{w})$

range: set of all outputs $T(\vec{v})$ $\quad C(A)$

kernel: set of all inputs for which $T(\vec{v}) = 0$ $\quad N(A)$

(7) Diagonalize $\rightarrow A = S\Lambda S^{-1}$

SVD for other cases: factor $A = U\Sigma V^T$

$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ $\quad AA^T = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad \begin{matrix} 0 & 0 \\ 0 & -6 \end{matrix}$

$\lambda_1 = 8, \quad \lambda_2 = 2 \quad \det = 16$

$u_1 = x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, u_2 = x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \begin{matrix} 6 & 0 \\ 0 & 0 \end{matrix}$

$A^TA = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

$\lambda_1 = 8: \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \lambda_2 = 2 \rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \rightarrow x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\frac{1}{\sqrt{2}} = \sqrt{2}$ $\quad v_1 = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \end{bmatrix}$

so $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $V = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{bmatrix}$

$\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$ $V^T = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{bmatrix}$

$\lambda_1 = 8 = \sigma_1^2$ so $\sigma_1 = \sqrt{8} = 2\sqrt{2}$
$\lambda_2 = 2 = \sigma_2^2$ so $\sigma_2 = \sqrt{2}$

Remember this:

the first r columns of V: rowspace of A
the last n-r columns of V: nullspace of A
the first r columns of U: column space of A
the last n-m columns of U: left nullspace of A

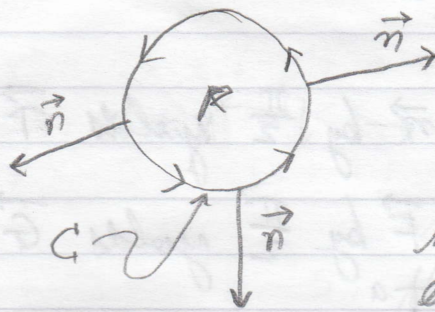
Check it out

$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$: $[\sqrt{2}\ \sqrt{2}] \rightarrow [2\ \ 2]$

128.1800 The afternoon is over, I will pick up from this point with #6 in S#058: Spectral Theorem. Now I will head over to the Commons for some chow, afterwhich I had better get back into M251 review.
I may start with $\vec{F} \cdot \vec{n}$ ← $\vec{n}$

and work backwards...
I will compile the review right here in my logbooks for posterity.

1 28. 2000 I am downloading Netscape Communicator 4.73. I will use it along with Netscape 6 PR1. I want to brush up on my intuitive understanding of what normal vector $\vec{n}$ represents in $\vec{F} \cdot \vec{n}$:



$\vec{n}$ = "outward pointing normal"

$\vec{F} = P\vec{i} + Q\vec{j}$

The field $\vec{F}$ and the region R have nothing to do with each other.

$\vec{F} \cdot \vec{n}$ is a function on C

$\oint_C \vec{F} \cdot \vec{n}\, |ds|$ is much like Green's Theorem, but

$\vec{n}$ is up, whereas $\vec{T}$ is tangential.

The Divergence Theorem says that the line integral is the same as the area integral:

$$\oint_C \vec{F} \cdot \vec{n}\, |ds| = \iint_R P_x + Q_y\, dA$$

whereas Green's Theorem is $\oint \vec{F} \cdot \vec{T}\, |ds| = \iint_R Q_x - P_y\, dA$

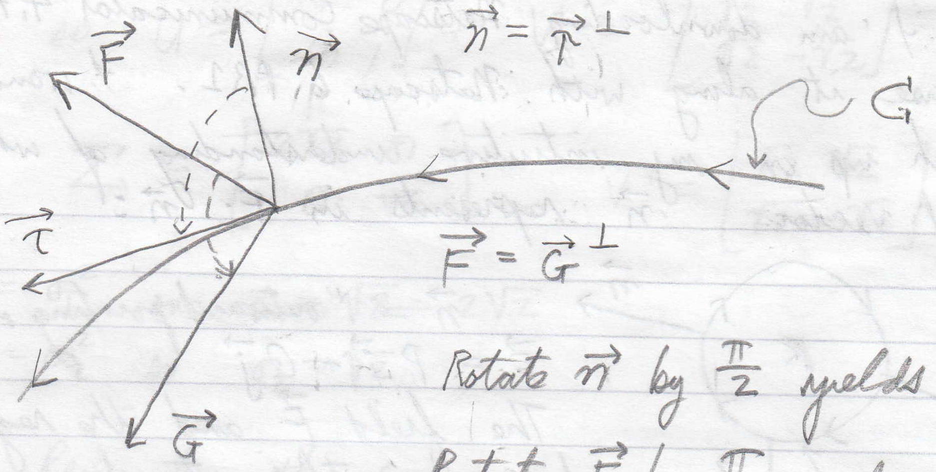
see this clearly $\vec{F} = P\vec{i} + Q\vec{j}$

$$\text{div}\, \vec{F} = P_x + Q_y$$

so $\oint \vec{F} \cdot \vec{n}\, |ds| = \iint_R \text{div}\, \vec{F} = \iint_R P_x + Q_y\, dA$

Now we will ZOOM IN on $\vec{n}$.

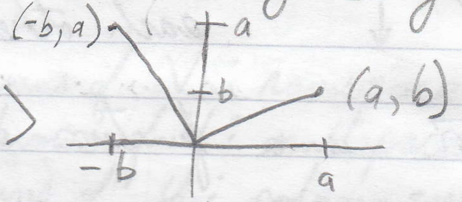
Look back on these notes one day. This is a very powerful idea!



Rotate $\vec{n}$ by $\frac{\pi}{2}$ yields $\vec{\tau}$ CCW

Rotate $\vec{F}$ by $\frac{\pi}{2}$ yields $\vec{G}$ CCW

$\vec{v} = \langle a, b \rangle$



(a,b) becomes $(-b,a)$ when rotated by $\frac{\pi}{2}$

$$\langle a, b \rangle \cdot \langle -b, a \rangle = -ba + ba = 0$$

$$\vec{F} \cdot \vec{n} = \vec{G} \cdot \vec{\tau} = \vec{F}^{\perp} \cdot \vec{\tau}$$

if $\vec{F} = P\vec{i} + Q\vec{j}$

$$\vec{G} = \vec{F}^{\perp} = -Q\vec{i} + P\vec{j}$$

$$\therefore \oint_C \vec{F} \cdot \vec{n} \, |ds| = \oint_C \vec{F}^{\perp} \cdot \vec{\tau} \, |ds|$$

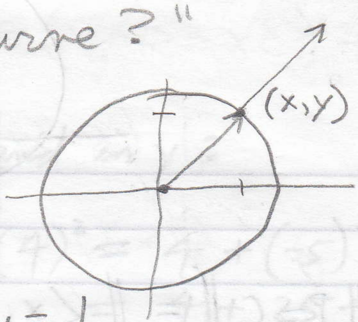
$$= \iint_R (Q_x - P_y)^{\perp} \, dA$$

$$= \iint_R +P_x + Q_y \, dA$$

Green's Theorem $\oint \vec{F} \cdot \vec{\tau} \, |ds| = \iint_R Q_x - P_y \, dA$

To find $\vec{n}$, ask yourself: "What is $\vec{n}$ at the point (x,y) on the curve?"

Consider the unit circle:



$$x^2+y^2=1 \rightarrow ax+by=1$$

$$\vec{n} = \langle a, b \rangle$$

Here $a=x$, $b=y$

so $\vec{n} = \langle x, y \rangle = x\vec{i} + y\vec{j}$

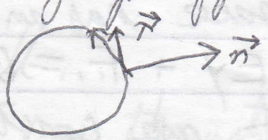
$= \langle \cos\theta, \sin\theta \rangle$

The field and the region have nothing to do with each other.

See session 127.2245 on pages 38–43 * to clarify. Now is time to move backwards to Line Integrals and then an all out review of Vector Calculus Concepts. Come midnight tomorrow night I will relax and take a deep breath. Come Tuesday evening, after work, I will be focused on Linear Algebra and on Linear Algebra alone. I will keep my review notes right here in logbook #63: ZONE 2: RIGHT HERE, RIGHT NOW.

** Go back to page 9 when this session ends.

128.2100 I am going off on a little "tangent" here caused by



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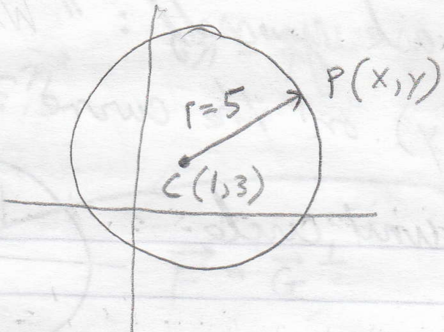
There is something extremely spiritual about this as I am referencing my old text book from my senior year at CBA (AP Calculus), ~1984



$$\| P - C \| = \| \langle x, y \rangle - \langle 1, 3 \rangle \| = \| \langle x-1, y-3 \rangle \|$$

$$= \sqrt{(x-1)^2 + (y-3)^2}$$

so P is on the circumference if $\sqrt{(x-1)^2+(y-3)^2} = 5$

$$(x-1)^2 + (y-3)^2 = 25$$

So in $x^2+y^2=1$, the center $(h,k) = (0,0)$

$$\text{circle} = \{ (x,y) : (x-h)^2 + (y-k)^2 = r^2 \}$$

Using the BINOMIAL THEOREM we can express $(x-1)^2 + (y-3)^2 = 25$

as $x^2+y^2-2x-6y-15=0$

Why? $(x-1)^2 = x^2-2x+1$

$(y-3)^2 = y^2-6y+9$

$(x^2-2x+1)+(y^2-6y+9) = 25$

becomes $x^2+y^2-2x-6y-15=0$

Every circle in 2-space has an equation of the form $x^2+y^2+Dx+Ey+F=0$

where D, E, and F are constants in $\mathbb{R}$.

In the reverse:

$$x^2+y^2-10x+8y-4=0$$

"complete the square" in x and in y:

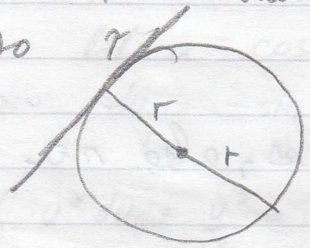
$$x^2-10x+(-5)^2+y^2+8y+(4)^2 = 4+(-5)^2+(4)^2$$
$$= 4+25+16$$
$$(x-5)^2+(y+4)^2 = 45$$

means: $(x-5)^2+(y-(-4))^2 = (\sqrt{45})^2 = (3\sqrt{5})^2$

So center is $(5,-4)$ and radius is $3\sqrt{5}$

... and now for the tangent $\vec{r}$ of a circle

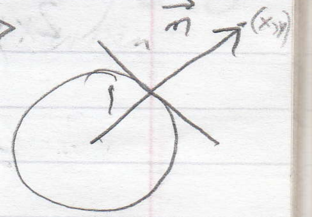
The tangent $\vec{r}$ to a circle is perpendicular to the diameter containing the point of contact: (x,y), so



If $\vec{r}$ is not a vertical or horizontal line, its slope is the negative reciprical of the slope of the diameter.

The distance from the center to the tangent is the radius.

So why is the normal vector $\langle x,y \rangle$ to $R = x^2+y^2=1$?



Oh well...

Let us run through some line integral problems.

Find a function f so that $df = (x+3y)\,dx + (3x - \sin y)\,dy$.

$$\int (x+3y)\,dx = \frac{x^2}{2} + 3yx + \gamma(y)$$

$$\frac{\partial}{\partial y}\left[\frac{x^2}{2} + 3yx + \gamma(y)\right] = 0 + 3x + \gamma'(y)$$

$$(3x - \sin y) = 3x + \gamma'(y)$$

$$\therefore \quad \gamma'(y) = -\sin y$$

$$\int -\sin y\,dy = \cos y + k$$

Hence $f(x,y) = \dfrac{x^2}{2} + 3yx + \cos y + k$

Is $x^2\,\vec{i} + xy\,\vec{j}$ a gradient?

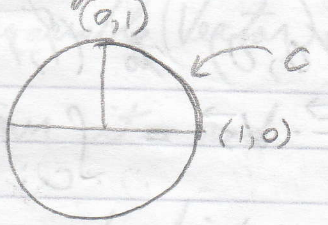
let $f_x = x^2$ and $f_y = xy$
does $f_{xy} = f_{yx}$? If so, yes, else no.

$f_{xy} = 0 \neq f_{yx} = y \quad \therefore$ NO

How about $xy^2\,\vec{i} + x^2y\,\vec{j}$?

$2xy = 2xy$ Yes.

Find $\int_C x^2\,dx + y^2\,dy$ where C is the first quadrant of $x^2+y^2=1$ from $(1,0)$ to $(0,1)$. 57



$(1,0)$ to $(0,1) \longrightarrow$ $x = 1-t$ $\quad \frac{dx}{dt} = -1\,dt$

$y = 0$ $\quad \frac{dy}{dt} = 0$

$$\int_0^1 (1-t)^2(-dt) + 0\,dt = \frac{(1-t)^3}{3}(-t)\Big|_0^1 = 0$$

Suppose I used polar coordinates

$$\int_0^{\pi/2} \left(\cos^2\theta(-\sin\theta) + \sin^2\theta(\cos\theta)\right) d\theta$$

let $u = \cos\theta$

$du = -\sin\theta\, d\theta$

$\int u^2\,du = \frac{u^3}{3} = \frac{\cos^3\theta}{3}$

let $u = \sin\theta$

$du = \cos\theta\, d\theta$

$\int u^2\,du = \frac{u^3}{3} = \frac{\sin^3\theta}{3}$

$$\left(\frac{\cos^3\theta}{3} + \frac{\sin^3\theta}{3}\right)\Big[_{\theta=0}^{\theta=\pi/2}$$

$\cos\frac{\pi}{2} = 0$

$\cos 0 = 1$

$\sin\frac{\pi}{2} = 1$

$\sin 0 = 0$

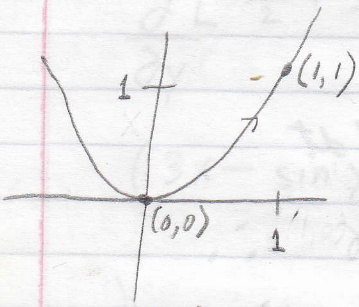
$$= \left(\frac{0}{3} + \frac{1}{3}\right) - \left(\frac{1}{3} + 0\right) = 0$$

the same answer either way.

Let $\vec{F} = y\,\vec{i} + x\,\vec{j}$

and C be curve running along parabola $y = x^2$ from $(0,0)$ to $(1,1)$.

Find $\int_C \vec{F} \cdot d\vec{s} = \int_C \vec{F} \cdot \vec{T}\,|ds|$



$x = t \qquad \frac{dx}{dt} = 1\,dt$

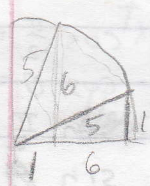
$y = t^2 \qquad \frac{dy}{dt} = 2t\,dt$

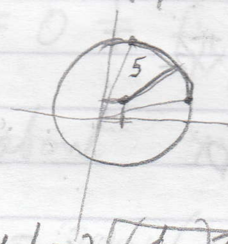
$$\int_C \vec{F} \cdot d\vec{s} = \int_{t=0}^{t=1} t^2(1)\,dt + t(2t)\,dt$$

$$= \int_0^1 \left(t^2 + 2t^2\right) dt = \int_0^1 3t^2\,dt$$

$$= t^3 \Big|_0^1 = 1$$

What about this one? Let C be the shorter arc connecting $(6,1)$ to $(1,6)$ on the circle $(x-1)^2 + (y-1)^2 = 25$. Find $\int_C x^2 + y\,|ds|$





How do I parametrize this? What is the value of θ at the two endpoints of the arc?

$$|ds| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{25} = 5$$

$x = 6 - 5t \qquad \frac{dx}{dt} = -5\,dt$

$y = 1 + 5t \qquad \frac{dy}{dt} = 5\,dt$

$$\int_C x^2 + y\,|ds| = 5\int_0^1 (6-5t)^2 + (1+5t)\,dt$$

I would like to take the scenic route with this one. After all, this could be my last night this into Multivariable (Vector) Calculus for a long time. I want to savor it and t5 Volume One is full.

Two paths:

First Path: expand first.

$$5\int_{t=0}^{t=1} \left(36-60t+25t^2+1+5t\right)dt$$

$$= 5\int_0^1 25t^2-55t+37\ dt$$

$$= 5\left[\frac{25}{3}t^3-\frac{55}{2}t^2+37t\right]_{t=0}^{t=1}$$

$$= 5\left(\frac{25}{3}-\frac{55}{2}+37\right) = 5\left(\frac{50}{6}-\frac{165}{6}+\frac{222}{6}\right)$$

$$= 5\left(\frac{107}{6}\right) = \frac{535}{6}$$

$37 \times 6 = 222$ $\quad 30 \times 6 = 180$ $\quad 272 - 165 = 107$

What happens when I do not expand? I predict an easier computation.

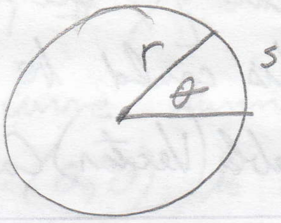
$$5\int_0^1 (6-5t)^2+(1+5t)\ dt$$

$$= 5\left[\frac{(6-5t)^3}{3}+t+\frac{5}{2}t^2\right]_0^1 = 5\left[\left(\frac{1}{3}+1+\frac{5}{2}\right)-\left(\frac{6^3}{3}\right)\right]$$

$$= 5\left(\left(\frac{2}{6}+\frac{6}{6}+\frac{15}{6}\right)-72\right) = 5\left(\frac{23}{6}-\frac{432}{6}\right)$$ NO WAY

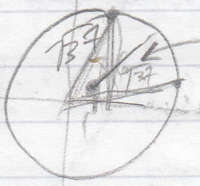
I will check it out with the computer, using DERIVE.

$\frac{535}{6}$ both ways

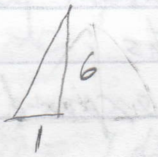


$s = r\theta$ radians

$\theta = \frac{s}{r}$

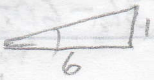


What is θ



$\tan\theta = 6$

$\arctan 6 = \theta = 1.4$ rad



$\arctan\left(\frac{1}{6}\right) = 0.165$

let $x = 5\cos\theta \quad dx = -5\sin\theta$

$\quad y = 5\sin\theta \quad dy = 5\cos\theta$

so $\theta = 1.4 - 0.165$

$\theta = 1.235$

$s = 6.175$

$$|ds| = \sqrt{25\sin^2\theta + 25\cos^2\theta}$$

$$= \sqrt{25} = 5$$

$$5\int_{\theta=0.165}^{\theta=1.4} (5\cos\theta)^2 + 5\sin\theta \, d\theta$$

$$= 5\int_{0.165}^{1.4} 25\cos^2\theta + 5\sin\theta \, d\theta$$

$$= 25\int_{0.165}^{1.4} 5\cos^2\theta + \sin\theta \, d\theta$$

$$= 25\int_{0.165}^{1.4} \frac{5}{2} + \frac{5\cos(2\theta)}{2} + \sin\theta \, d\theta$$

$$= 25\left(\frac{5\theta}{2} + \frac{5\sin(2\theta)}{4} - \cos\theta\right)\Big[_{0.165}^{1.4}$$

use derive: 97.9

but $\frac{535}{6} = 89.6$

I will use rectangular coordinates.

$\boxed{\begin{array}{l}2000.129.1\\05.08.0115\end{array}}$ Still going strong. I can sleep all day tomorrow before the final Calc 3 exam. After that, this semester is really almost over; but then I will focus 100% on Linear Algebra for 2 days.

Now for a review of "ancient concepts from January 2000 to April —

No order here. Just random bits.

cylindrical: $x = r\cos\theta$

$y = r\sin\theta$

element $dV = r\,dr\,d\theta\,dz$

spherical: think $r \leftrightarrow \rho\sin\phi$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

vector differentiation: $\frac{d}{dt}[\vec{u}\cdot\vec{v}] = (\vec{u}'\cdot v) + (\vec{u}\cdot\vec{v}')$

$\frac{d}{dt}(\vec{u}\times\vec{v}) = (\vec{u}'\times\vec{v}) + (\vec{u}\times\vec{v}')$

if $(\vec{u}\times\vec{v})\times\vec{w}$, just apply it twice.

see Sketch Diary

Gradient $\vec{\nabla} g = \langle f_x, f_y, f_z\rangle$

Divergence $\nabla F = \operatorname{div}\vec{F} = P_x + Q_y + R_z$

$\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}$

$\vec{x} = p\vec{u} + q\vec{v} + r\vec{w}$

Think: $p\left[\vec{u}\right] + q\left[\vec{v}\right] + r\left[\vec{w}\right] = \left[\vec{x}\right]$

Think $\perp$ rref $\left(\left[\vec{u}\ \vec{v}\ \vec{w}\ \vec{x}\right]\right)$

yields $\begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \end{bmatrix}$

$\vec{u}, \vec{v}, \vec{w}$ are given in the form
$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$
$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$
$\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$
$\vec{x} = x_1\vec{i} + x_2\vec{j} + x_3\vec{k}$

Just set them up

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = p\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + q\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + r\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

and solve $\begin{bmatrix} u_1 & v_1 & w_1 & x_1 \\ u_2 & v_2 & w_2 & x_2 \\ u_3 & v_3 & w_3 & x_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \end{bmatrix}$

For $\vec{w}$ to exist in $\vec{u} \times \vec{w} = \vec{v}$
test: if $\vec{u} \cdot \vec{v} = 0$, then let $\vec{w} = \langle a, b, c \rangle$
Do $\vec{u} \times \vec{w}$ and solve for $\langle a, b, c \rangle$.
If $\vec{u} \times \vec{v} \neq 0$, $\vec{w}$ DNE.

$\perp$ distance from r_0 to plane $ax + bx + cz = d$
$\vec{n} \cdot \vec{r} = d$ DIST $= |\vec{n} \cdot (\vec{r} - \vec{r}_0)| / |\vec{n}|$

$$\frac{|\vec{n} \cdot \vec{r} - \vec{n} \cdot r_0|}{|\vec{n}|} = \frac{|d - \vec{n} \cdot \vec{r}_0|}{|\vec{n}|}$$

Let $\vec{u}(t) = \langle 4 + e^{3t}, 3 - 2e^{3t}, 4 - e^{3t} \rangle$

What is the length of the trajectory followed from $t=0$ to $t=2$?

$$L = \int_0^2 \left| \vec{u}\,'(t) \right| dt = \int_0^2 \sqrt{(3e^{3t})^2 + (6e^{3t})^2 + (3e^{3t})^2}$$

$$= \int_0^2 \sqrt{(9+36+9)(e^{3t})^2}\, dt \qquad , (e^{3t})^2 = e^{6t}$$

$(x^3)^2 = x^6$

$(e^{3t}) = e^{6t}$

$$= \int_0^2 \sqrt{54}\, e^{3t}\, dt = 3\sqrt{6} \int_0^2 e^{3t}\, dt$$

$$= 3\sqrt{6} \left(\tfrac{1}{3} e^{3t} \right) \Big|_0^2 = \sqrt{6}\left(e^6 - 1\right)$$

X

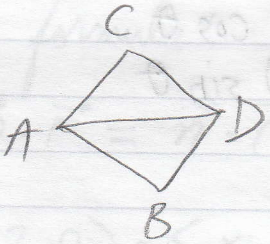
Directional Derivative of f determined by $\vec{\nabla} g$.

let $\vec{u} = \dfrac{\vec{\nabla} g}{\|\vec{\nabla} g\|}$ $\qquad D_{\vec{u}} f = \vec{\nabla} f \cdot \dfrac{\vec{\nabla} g}{\|\vec{\nabla} g\|}$

where $\vec{\nabla} g = \langle g_x, g_y, g_z \rangle$

X

Area



$\dfrac{\|\vec{a} \times \vec{b}\|}{2} = \text{area of } \Delta$

X

Plane passing through P_0 and P_1 and $\perp$ to plane $ax + by + cz = d$,

let $\vec{n}_1 = \langle a, b, c \rangle$ let $\vec{v} = P_0 - P_1$

then $\vec{n}_1 \times \vec{v} = \vec{n}$

then $\vec{n} \cdot (\vec{r} - r_0) = 0$ where $r_0 = P_0$ or P_1

And that's all there is 'cause there aint no more...

Don't complicate things.

$$\iiint_S x+2y+3z \, dV$$

I will have to finish this session tomorrow when I rise. I will go over a few problems from the second midterm, and then, I will relax and go through this and t∑.

129. 1045 Now, where was I? Forgot that one. Levitt may have made a mistake

Do

$$\iiint_S x^2+y^2+z^2 \, dV$$

where $x^2+y^2+z^2=\rho^2$ above z

$z = r \cos\theta$

$r = r \sin\rho$

$x = r \cos\theta = \rho \sin\phi \cos\theta$

$y = r \sin\theta = \rho \sin\phi \sin\theta$

$0 \leq \theta \leq 2\pi$

$0 \leq \rho \leq 1$

$0 \leq \phi \leq \pi/4$

$$\int_0^{2\pi}\int_0^1\int_0^{\pi/4} \rho^2 \cdot \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta$$

I will not solve this. The WORK will do itself. Setting the problem up is everything.

Some concepts from February:

Find an equation of the plane that passes through the line of intersection of the planes $x+y+z=1$ and $x+y-z=2$ and that also passes through the point $(1,1,1)$.

SACRED THOUGHTS at this point are:

we have $\vec{r} = \langle x, y, z \rangle$

$\vec{r}_0 = \langle 1, 1, 1 \rangle$

$\vec{n}_1 \times \vec{v} = \vec{n}$ where $\vec{n}_1 = \langle 1,1,1 \rangle$, $x+y+z=1$

$\vec{v} = \langle 1,1,-1 \rangle$, $x+y-z=2$

$$\vec{n}_1 \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \cancel{-2-(-2)-0 = -2+2=0}$$

$$-2\vec{i} + 2\vec{j} + 0\vec{k}$$

$$\vec{n} = \langle -2, 2, 0 \rangle$$

sacred formula: $\vec{n} \cdot (\vec{r} - r_0) = \emptyset$

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

$$\langle -2, 2, 0 \rangle \cdot \langle x, y, z \rangle = \langle -2, 2, 0 \rangle \cdot \langle 1, 1, 1 \rangle$$

$$-2x + 2y = -2 + 2 = 0$$

I am going to do one more problem here, and then I will just go over all my notes for the rest of the day. There is a time limit (constraint). I have to study "mentally", "silently"...

Find the values for which the velocity vector of the curve $\vec{r} = \langle t, 1-t, t-t^3 \rangle$ is parallel to $\langle 1, -1, 0 \rangle$.

$$r'(t) = \langle 1, -1, 1-3t^2 \rangle$$

$$\langle 1, -1, 0 \rangle = \vec{c} \langle 1, -1, 1-3t^2 \rangle$$

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \vec{c} \begin{bmatrix} 1 \\ -1 \\ 1-3t^2 \end{bmatrix}$$

when $3t^2 = 1$, $t^2 = \frac{1}{3}$, $t = \pm \frac{1}{\sqrt{3}}$

How will I remember all this? There is much more in t4... right here, right now.

One note about $\vec{n}$: see page 19!

$$\vec{n} = \frac{\left\langle \frac{dy}{dt}, -\frac{dx}{dt} \right\rangle}{|ds|}$$

$F = xy^2 i + x^2 y j$ $\qquad \vec{n} = \dfrac{\langle 2xy, -2xy \rangle}{\sqrt{4x^2y^2 + 4x^2y^2}}$

$$\vec{n} = \frac{\langle 2xy, -2xy \rangle}{2xy\sqrt{2}}$$

???

I think I had better steer clear of long philosophic inquiries into the nature of $\vec{n}$.

Remember that $\oint \vec{F} \cdot \vec{n}\, |ds| = \iint_R \operatorname{div} \vec{F}\, dA$

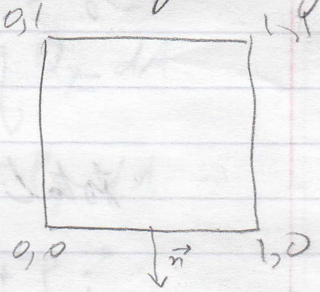
$= \iint_R P_x + Q_y \, dA$!!!

if $\vec{F} = \langle x, y^2 \rangle$

$\operatorname{div} \vec{F} = \langle 1, 2y \rangle$

From the sample problems: Verify the divergence theorem by computing $\oint \vec{F} \cdot \vec{n}\, |ds|$ where $F = x\vec{i} + y\vec{j}$

C is the four line segments bounding the square $D = \{ 0 \leq x \leq 1, 0 \leq y \leq 1 \}$



$\operatorname{div} \vec{F} = \langle 1, 1 \rangle$

$\int_{y=0}^{y=1} \int_{x=0}^{x=1} 2 \, dx\, dy = 2$

or c_1: $x = t \quad dx = 1$

$y = 0 \qquad 0 \leq t \leq 1$

$\vec{n} = -j$

~~$\int_0^1 xi(-j) + yj(-j)$~~

$\int_0^1 \langle x, y \rangle \cdot \langle 0, -1 \rangle = 0$

$\int_0^1 \langle x, y \rangle \cdot \langle 0, -1 \rangle dt$

$= \int_0^1 0 \, dt = \frac{t^2}{2} = 0$

c_2: $x = 1 \quad dx = 0$

$y = t \quad dy = 1$

$\vec{n} = +i$

$\vec{n}$ is $\langle \frac{dy}{dt}, -\frac{dx}{dt} \rangle$

$\int_0^1 \langle x, y \rangle \cdot \langle \frac{dy}{dt}, -\frac{dx}{dt} \rangle dt$

$= \int_0^1 \langle 1, t \rangle \cdot \langle 1, 0 \rangle dt = \int_0^1 1 \, dt = t \Big|_0^1 = 1$

C_3: $(1,1)$ to $(0,1)$

$$x = 1-t \quad dx = -1\,dt$$
$$y = 1 \quad dy = 0$$

$$\int_0^1 \langle x,y\rangle \cdot \left\langle \frac{dy}{dt}, -\frac{dx}{dt} \right\rangle dt$$

$$= \int_0^1 \langle 1-t, 1\rangle \cdot \langle 0, +1\rangle\, dt$$

$$= \int_0^1 1 \, dt = t\Big|_0^1 = 1$$

C_4: $(0,1)$ to $(0,0)$

$$x = 0 \quad dx = 0$$
$$y = 1-t \quad dy = -1\,dt$$

$$\int_0^1 \langle 0, 1-t\rangle \cdot \langle -1, 0\rangle\, dt$$

$$= \int_0^1 0$$

total: $0+1+1+0 = 2$

The Same!

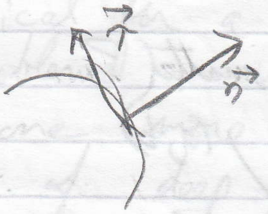
One more time — slowly, with comments

The clue came from notes I had taken at Borders Bookstore in East Brunswick last week. See p. 19:

$$\vec{n} = \frac{\frac{dy}{dt}\vec{i} - \frac{dx}{dt}\vec{j}}{\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(-\frac{dx}{dt}\right)^2}} = \frac{\left\langle \frac{dy}{dt}, -\frac{dx}{dt} \right\rangle}{|ds|}$$

So, $\oint_C \vec{F}\cdot\vec{n}\,|ds| \longrightarrow \oint_C \langle P, Q \rangle \cdot \langle dy, -dx \rangle$ dt

Think  $\vec{T} = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle$



$\vec{n} = \vec{T}^{\perp}$

see p. 51, 52

$$(a, b)^{\perp} = (-b, a)$$

$$\vec{T}^{\perp} = \left[\langle \frac{dx}{dt}, \frac{dy}{dt} \rangle\right]^{\perp} = \langle \frac{dy}{dt}, -\frac{dx}{dt} \rangle$$

So, that problem again: $\vec{F} = x\vec{i} + y\vec{j}$

$$\oint \vec{F}\cdot\vec{n}\,|ds| = \iint_R P_x + Q_y \, dA$$

notice $\oint \vec{F}\cdot\vec{T}\,|ds| = \iint_R Q_x - P_y \, dA$

$$\vec{F} = \underset{a}{P}\vec{i} + \underset{b}{Q}\vec{j}, \quad \vec{F}^{\perp} = \underset{-b}{-Q}\vec{i} + \underset{a}{P}\vec{j}$$

so $\iint_R P_x + Q_y \, dA \longleftarrow \oint \vec{F}\cdot\vec{n}\,|ds|$

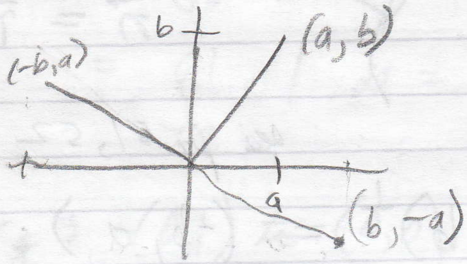
$-b$ $\qquad$ a

$\iint_R \underset{a}{Q_x} - \underset{b}{P_y} \, dA \longleftarrow \oint \vec{F}\cdot\vec{T}\,|ds|$

Why is $\perp$: $(a,b)^{\perp} = (-b, a)$

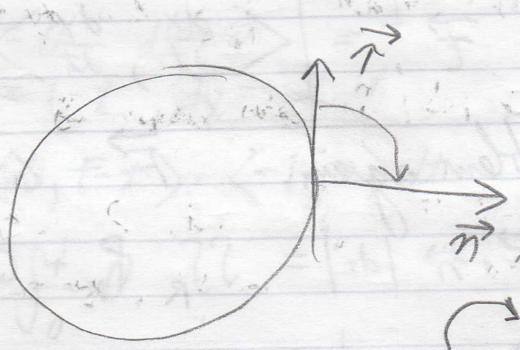
but $\vec{n} = \vec{\tau}^{\perp} = \dfrac{\langle \frac{dy}{dt}, -\frac{dx}{dt} \rangle}{|ds|}$?

Does it matter?



$(-b, a)$ $\frac{\pi}{2}$

$(b, -a)$ $-\frac{\pi}{2}$



$\vec{n}$ is in essence, the $-\frac{\pi}{2}$ type of rotation!

From $\vec{\tau}$ to $\vec{n}$: $(b, -a)$

From $\vec{n}$ to $\vec{\tau}$: $(-b, a)$

From $\oint \vec{F} \cdot \vec{\tau} \, |ds| = \iint_R Q_x - P_y \, dA$

to $\oint \vec{F} \cdot \vec{n} \, |ds| = \iint_R P_x + Q_y \, dA$

notice $\underset{a}{Q_x} + \underset{b}{(-P)_y} \longrightarrow \underset{-b}{P_x} + \underset{a}{Q_y}$

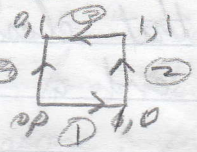
But in the opposite direction $\underset{a}{P_x} + \underset{b}{Q_y} \longrightarrow \underset{b}{Q_x} - \underset{-a}{P_y}$

129-1300 I will gladly explore this concept after my exams are finished, but now I have to have a firm grasp ON a computational level (practical for a student trying to solve exam problems). So, I will do this problem one more time with the minimal amount of deep understanding. Then I will meditate upon all my t's and L63 M251 notes.

$\vec{F} = \langle x, y \rangle$ $\quad D = \{0 \le x \le 1, 0 \le y \le 1\}$

First $\iint_R P_x + Q_y \, dA = \int_0^1 \int_0^1 2 \, dx \, dy = 2$

next $\oint \vec{F} \cdot \vec{n} \, |ds|$: for segments



① $x = t \quad dx = 1dt$ $\quad \int_0^1 \langle t, 0 \rangle \cdot \langle 0, -1 \rangle \, dt$
$y = 0 \quad dy = 0$
$= \int_0^1 0 \, dt = 0$

② $x = 1 \quad dx = 0$ $\quad \int_0^1 \langle 1, t \rangle \cdot \langle 1, 0 \rangle \, dt$
$y = t \quad dy = 1 \, dt$
$= \int_0^1 1 \, dt = t \Big|_0^1 = 1$

③ $x = 1 - t \quad dx = -1 dt$
$y = 1 \quad dy = 0$ $\quad \int_0^1 \langle 1-t, 1 \rangle \cdot \langle 0, 1 \rangle \, dt$
$= \int_0^1 1 \, dt = t \Big|_0^1 = 1$

④ $x = 0 \quad dx = 0$
$y = 1 - t \quad dy = -1 \, dt$ $\quad \int_0^1 \langle 0, 1-t \rangle \cdot \langle -1, -0 \rangle \, dt$
$= 0$

total = 2 ✓

129-1600 : $ note : balance 500
ins due 300
200 ok pay it!

let's try another one off the cuff. Now I am curious to see just how TAME or WILD most random symmetric matrices can get. Do they get out of hand without a computer?

$$A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \quad \begin{array}{l} \text{trace}(A) = 9 \\ \det(A) = 18 - 4 = 14 \end{array}$$

characteristic polynomial: $\lambda^2 - 9\lambda + 14 = 0$

proof: $(6-\lambda)(3-\lambda) - 4 = 18 - 9\lambda + \lambda^2 - 4$

$a = 1$

$b = -9$ $\quad \lambda = \dfrac{9 \pm \sqrt{81 - 56}}{2} = \dfrac{9 \pm \sqrt{25}}{2}$

$$\begin{array}{r} 81 \\ \underline{56} \\ 25 \end{array}$$

$c = 14$ $\quad = \dfrac{9 \pm 5}{2}$ $\quad \lambda_1 = \dfrac{14}{2} = 7$

$\lambda_2 = \dfrac{4}{2} = 2$

11 Very cool! The Quadratic Equation is beautiful.

$(\lambda - 7)(\lambda - 2) = 0$

$(A - \lambda_1 I)\vec{x} = 0 \qquad \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \vec{x}_1 = 0$ when $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ← b

← λ − a

= 1

$q_1 = \dfrac{\vec{x}_1}{\|\vec{x}_1\|} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$, $\quad q_2 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

$(A - \lambda_2 I)\vec{x} = 0 \qquad \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \vec{x}_2 = 0$ when $\vec{x}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\begin{array}{l} 2-3 \\ \leftarrow \lambda_2 - c \\ \leftarrow b \end{array}$

My FAITH in the power of math is restored.

$S = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ but $Q = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

$$A = Q\Lambda Q^T = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

Note that $Q\Lambda Q^T = Q\Lambda Q^{-1}$ and $\Lambda = Q^{-1} A Q$

This stuff is so cool. I dont want to forget it. I know it is tempting to just put aside material after a semester is over; but — unlike Multivariable Calculus, as a computer scientist, Linear Algebra will prove to be very useful.

For any 2×2 symmetric matrix $\det(A-\lambda I) =$

$$\lambda^2 - \underbrace{\text{TRACE}}_{a+c}(\lambda) + \underbrace{\text{DET}}_{ac-b^2} = 0$$

The test for real roots of $Ax + Bx + C = 0$ is based on $B^2 - 4AC$.
This must NOT be negative or else its square root in the quadratic formula would be imaginary. (IMAGINARY NUMBERS are generally evil).

Our equation, $\lambda^2 - T\lambda + D = 0$, so the test is based on $T^2 - 4D =$
$(a+c)^2 - 4(ac-b^2)$ must not be negative.

This is awesome: Expand $(a+c)^2 - 4(ac-b^2)$
$a^2 + 2ac + c^2 - 4ac + 4b^2 = a^2 - 2ac + 4b^2 + c^2$
Recollect terms as $(a-c)^2 + 4b^2$
This cannot be negative so the ROOTS λ_1 and λ_2 must be real!
$(a-c)^2$ is always nonnegative, as is b^2.

and the eigenvectors are perpendicular: $\vec{x}_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix}$

$\vec{x}_2 = \begin{bmatrix} \lambda_2 - c \\ b \end{bmatrix}$

What's up with POSITIVE DEFINITE Matrices?

Symmetric matrices $A = A^T$ with all positive eigenvalues are "positive definite".

$$ax^2 + 2bxy + cy^2 = \underset{\uparrow \atop \text{pivot}}{a}\left(x + \frac{b}{a}y\right)^2 + \underset{\uparrow \atop \text{pivot}}{\left(\frac{ac-b^2}{a}\right)} y^2$$

Positive pivots give a positive definite matrix.

Think back to the factorization $A = LDL^T$

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & \frac{ac-b^2}{a} \end{bmatrix}\begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}$$

DIG THIS → Elimination follows the same steps as "completing the square"

It deals with the first column, and fixes the rest later. The numbers that come out are identical!

Outside the squares are the pivots.
Inside $(x + \frac{b}{a}y)^2$ are the numbers 1 and $\frac{b}{a}$ from L. Every positive definite matrix factors into $A = LDL^T$ with positive pivots.

(11) next: A^{-1} use elimination on augmented matrix

$$[A \ I] \longrightarrow [I \ A^{-1}]$$

or find C_A, then C_A^T and $\det A$

$$A^{-1} = \frac{C^T}{\det(A)}$$

(12) Applications to Differential Equations

$$\frac{d\vec{u}}{dt} = \lambda \vec{u} \text{ has solutions } \vec{u}(t) = \vec{c}\, e^{\lambda t}$$

Suppose we found eigenvalues of A

$\left.\begin{array}{l}\lambda_1 = 1\\ \lambda_2 = 2\\ \lambda_3 = 3\end{array}\right\}$ solve $\frac{d\vec{u}}{dt} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \vec{u}$ starting from

$\vec{u}(0) = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$ Step 1: eigenvectors

$$\left.\begin{array}{l}
\lambda_1 = 1 : \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} ; \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\[2ex]
\lambda_2 = 2 : \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} ; \quad \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\[2ex]
\lambda_3 = 3 : \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} ; \quad \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{array}\right\} S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 2: solve for $\vec{c}$: $S\vec{c} = \vec{u}(0)$

My way: $\text{rref}(S^T[S\vec{c}])$

$$\underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \underset{3\times 4}{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 4 \end{array}\right]} = \underset{3\times 4}{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 2 & 11 \\ 1 & 2 & 3 & 15 \end{array}\right]} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 2 & 9 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 4 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right] ; \quad \vec{c} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

of course, but they won't all be so easy; hence, the WORK.

$$\vec{u}(t) = c_1 e^{\lambda_1 t}\vec{x}_1 + \dots + c_n e^{\lambda_n t}\vec{x}_n$$

$$\vec{u}(t) = e^{t}\begin{bmatrix}1\\0\\0\end{bmatrix} + e^{2t}\begin{bmatrix}1\\1\\0\end{bmatrix} + 4e^{3t}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Second order equations

$y'' + by' + ky = 0$ let $y = e^{\lambda t}$

$y' = \lambda e^{\lambda t}$

$y'' = \lambda^2 e^{\lambda t}$

$e^{\lambda t}(\lambda^2 + b\lambda + k) = 0$

It all depends on $\lambda^2 + b\lambda + k = 0$

$$\frac{d}{dt}\begin{bmatrix}y\\y'\end{bmatrix} = \begin{bmatrix}0 & 1\\-k & -b\end{bmatrix}\begin{bmatrix}y\\y''\end{bmatrix},$$

$\vec{x}_1 = \begin{bmatrix}1\\\lambda_1\end{bmatrix}$

$\vec{x}_2 = \begin{bmatrix}1\\\lambda_2\end{bmatrix}$

$\vec{u}(t) = c_1 e^{\lambda_1 t}\begin{bmatrix}1\\\lambda_1\end{bmatrix} + c_2 e^{\lambda_2 t}\begin{bmatrix}1\\\lambda_2\end{bmatrix}$

$-A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$ is stable

and $\vec{u}(t) \to 0$ when ① $\text{trace}(A) = a+d < 0$

② $\det(A) = ad - bc > 0$.

$\boxed{\begin{matrix}2000.131.3\\05.10.01.00\end{matrix}}$ I will pick up with the Matrix exponential tomorrow morning (Done p47). When I finish these review sessions in L63, I will go to bank (deposit 150, keep 50), pay fines, and go to probation officer.

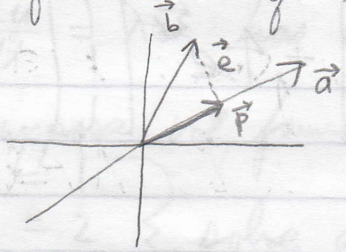
When I return, for the entire evening, I will simply go through L63, then tS, and then L62, and even L61.

NOTE: JUST GO THROUGH tS !!!

and LOGBOOKS

131.1045 page 120 of L61 has a nutshell for the foundation of Gram Shmidt.

13



$\vec{p} = \vec{a}x$

$\vec{e} = (\vec{b} - \vec{p}) = \vec{b} - \vec{a}x$

$\vec{a} \perp \vec{e}$ so $\vec{a}^T \vec{e} = 0 = \vec{a}^T(\vec{b} - \vec{a}x)$

$\vec{a}^T\vec{b} - x\vec{a}^T\vec{a} = 0$

$\vec{a}^T\vec{b} = x\vec{a}^T\vec{a}$

$x = \dfrac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}$ $\qquad \vec{p} = \vec{a}x = \vec{a}\,\dfrac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}$

the projection matrix can be extracted by seeing what is acting on $\vec{b}$ to project it onto $\vec{a}$ as $\vec{p}$: $\vec{p} = P\vec{b}$

so rewrite $\vec{p} = \vec{a}x = P\vec{b} = \dfrac{\vec{a}\vec{a}^T}{\vec{a}^Ta}\vec{b}$

and $P = \dfrac{\vec{a}a^T}{\vec{a}^T\vec{a}}$

Note: If vectors are arranged as the columns of a square matrix, then they are linearly independent iff the determinant of that matrix is not zero.

Is the projection matrix symmetric?

$(\vec{a}\vec{a}^T)^T = \vec{a}^{T^T}\vec{a}^T = \vec{a}\vec{a}^T$, yes $P = P^T$

$A\hat{x} = \vec{p}$

$$p = \hat{x}_1 \vec{a}_1 + \hat{x}_2 \vec{a}_2$$

We want these in matrix form: $A^T\vec{e} = 0$

$$A^T(\vec{b} - A\hat{x}) = 0 \Rightarrow A^T\vec{b} - A^TA\hat{x} = 0$$

$A^TA\hat{x} = A^T\vec{b}$; $A^T\vec{e} = 0$

remember, $\vec{e} = \vec{b} - A\hat{x}$

$\vec{e} = \vec{b} - A\hat{x} \in N(A^T)$.

$\hat{x} = (A^TA)^{-1}A^Tb$; $\vec{p} = A\hat{x} = Pb$

where $P = A(A^TA)^{-1}A^T$

(14) For least squares: (t, b) $A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \end{bmatrix} \underset{x}{\begin{bmatrix} C \\ D \end{bmatrix}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

use MAGIC FORMULA $A^TA\hat{x} = A^T\vec{b}$

$\text{rref}\left(\begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \end{bmatrix} \left[\begin{array}{cc|c} 1 & t_1 & b_1 \\ 1 & t_2 & b_2 \\ 1 & t_3 & b_3 \end{array}\right] \right)$ to find C, D ; $\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix}$

If we orthonormalize the column vectors:

$A^TA\hat{x} = A^T\vec{b}$ becomes $Q^TQ\hat{x} = Q^Tb$

$\hat{x} = Q^T\vec{b}$ because $Q^TQ = I$

$$\vec{x}_i = \vec{q}_i^{\,T}\vec{b} = \vec{q}_i \cdot \vec{b}$$

$A = LU \longrightarrow A = QR$ $R = Q^{-1}A = Q^TA$

(15) When using Gram Schmidt $w_3 = v_3 - \frac{w_2^T v_3}{w_2^T w_2} w_2 - \frac{w_1^T v_3}{w_1^T w_1} w_1$

$q_3 = \frac{w_3}{\|w_3\|}$

$\vec{p} = (q_1 \cdot b) q_1 + \dots + (q_n \cdot b) q_n$

(16) Cramer's Rule $x_j = \frac{\det(B_j)}{\det(A)}$

$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\det(A)}$ (example)

(17) area of triangle $= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

if $\vec{v} = (3,2)$ $\vec{w} = (1,4)$ → 10

area $= \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = \|v \times w\|$

$v \times w = \begin{vmatrix} i & j & k \\ 3 & 2 & 0 \\ 1 & 4 & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + 10\vec{k}$

$\|\langle 0, 0, 10 \rangle\| = \sqrt{100} = 10$

(18) Algorithm for $A\vec{x} = b$ when more equations than unknowns (more rows than columns, $m > n$, "tall and thin"): $A^TA\hat{x} = A^T\vec{b}$

$\text{rref}(A^T[A\ b]) \longrightarrow \hat{x}$; then $p = A\hat{x}$ $P = A(A^TA)^{-1}A^T$

$b - p = e$ so $e^Tb = 0$ and $e^Tp = 0$

$\|e\| \rightarrow$ distance from b to p.

If C: becomes corrupt, this would be a very useful alternative to a Full restore to original factory systems; but I see no "Typical restore" option.

So I will perform another systems save and recreate the restore diskette.

I would hate to start over at this point. If this diskette would do what it says it can do in the documentation, then this would be a great solution to a corrupt, fatal error on C:

After the systems save, I will try one more time to make the diskette.

I am actually tired before midnight. I will finish tomorrow.

$\boxed{\begin{array}{l}2000\text{-}140.5\\05.19.0100\end{array}}$

Multivariable Calculus	640:251	A	4.0
Linear Algebra	640:250	B+	3.5
Mathematical Reasoning	640:300	B+	3.5

$3.66 \approx 3.7$

$$\begin{array}{r} 3.66 \\ 3\overline{)11.0} \\ \underline{9} \\ 20 \\ \underline{18} \\ 20 \end{array}$$

Behold page 97 (L63 this)
This was the one missing!
I did better than I had hoped!
Also see p. 33 (L63 this)
p. 4 (L63)

see p. 141–143 (L62)
p. 136, 137 (L62)
p. 126 (L62)

L61: p 141 + 142 see list of pages there.

$

$$\begin{array}{rl} 300 & \\ -\ 60 & \text{phone} \\ \hline 240 & \\ -\ 40 & \text{cat} \\ \hline 200 & \\ \text{dep } 200 & \\ \hline 400 & \end{array}$$

The solution to the problem of caring about what others think of oneself is found in Schopenhauer's WISDOM OF LIFE, Position, or A Man's Place in the Estimation of Others.

I will reread this and develop an Inner Space that will transcend the opinions of others. The opinions of my sister and her husband (about me) are not essential to my happiness.

I quote: "... what goes on in other people's consciousness is a matter of indifference to us; and in time we get really indifferent to it, when we come to see how superficial and futile are most people's thoughts, how narrow their ideas, how mean their sentiments, how perverse their opinions, and how much of error there is in most of them; when we learn by experience with what depreciation a man will speak of his fellow, when he is not obliged to fear him, or thinks that what he says will not come to his ears. And if ever we have had an opportunity of seeing how the greatest of men will meet with nothing but slight from half-a-dozen blockheads, we shall understand that to lay great value upon what other people say is to pay them too much honor."

So, I will no longer honor my sister or her husband. They do not deserve my honor. What they say has no value to me. Likewise, I do not honor what anyone says about me, be it good or ill.